

Schrödinger Equation and Phase Space in Quantum Mechanics

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Abstract

Using classical statistics, Schrödinger equation in quantum mechanics is derived from *complex space* model[4]. Phase-space probability amplitude, that can be defined on classical point of view, has connections to probability amplitude in internal space and to wave function in quantum mechanics. In addition, the physical entity of wave function in quantum mechanics is confirmed once again.

1 Introduction

Since E. Wigner and L. Szilard proposed the phase-space distribution [2][5] to study quantum corrections to the classical statistical mechanics in 1932, the distribution (*Wigner Distribution*) has been studied in quantum optics, quantum chemistry, and *etc.* In 1947, Moyal, J. E.[3] attempted to interpret quantum mechanics as a statistical theory, and some other people have tried it with the distribution or using some other way. Especially, L.S.F. Olavo made a bridge from classical statistics to quantum theory using the transformation, *Wigner-Moyal* Infinitesimal transformation¹ and Liouville's theorem in classical phase space[6][1]. It was a brilliant insight in physics. But physical meaning of the expressions, especially ontological point of view, has not been interpreted yet.

¹the name was cited by L.S.F. Olavo.

In the paper, *The Wave Function in Quantum Mechanics*[4], we supposed that physical space consists of *4-dimensional complex space* and that there are interactions among real physical object and *vacuum particles*(negative energy mass electrons). Also we assumed that electromagnetic energy is propagated through vacuum-particle-string oscillations.

In those assumptions, words, *intangible* and *imaginary world* were used because we can not prove the existence directly through experiments and also because we have not recognized the interactions with *vacuum particles*. Even in the definition of word, *phenomena*, we have not included or connected to those particles(not antiparticles). Being considered that the *complex space* was supposed not for a mathematical tool, but for a genuine physical space in ontology, those intangible particles in imaginary world should be, ultimately, included in phenomena through a physical conception.

In former paper[4] ; first, Plank's constant(h) was investigated in special theory of relativity, then the physical meaning of *photon* was interpreted. Finally, the physical entity of wave function in quantum mechanics was searched through new interpretation of special theory of relativity. Furthermore, we could have a clue which can explain *the second law of thermodynamics* if we consider that the interaction with vacuum particles is unavoidable, thus a physical system cannot be isolated from vacuum particles. However, one of conclusions is following : When physical object has kinetic energy, it is wrapped, in the respect of phenomena, with vacuum particles (vacuum electrons in the model), those of which are interacting with the physical object; the effect of interaction is following the physical object in a wave form through vacuum-particle-string oscillations (*i.e.* transverse mode) with velocity v_g , that is the same as that the physical object has. In addition, those vacuum particles, in fact, carry the physical momentum p , that is the momentum of the physical object conventionally and also they have kinetic energy, pc .

Even though the physical entity of wave function in quantum mechanics was interpreted as a representation of interaction with vacuum particles wrapping and following the physical object in *phenomenological view*, actually the interaction propagates in an imaginary subspace.

In Section 2, Schrödinger equation is derived using classical statistics or fundamental statistical concept. In the process, we can find that phase space representation is also possible, definitely, thus in the formalism, it cannot be discriminated in the name of uncertainty principle. In addition, the physical meaning of *Wigner-Moyal* Infinitesimal transformation or characteristic function[1] is interpreted with classical statistics and complex space model. Finally, Summary is followed.

2 Classical Statistics

2.1 classical phase-space and ensemble

In classical phase space, the physical meaning of an *ensemble* is a geometrical point set (mental copies of a system), each point of which is corresponded to a same, can not be distinguished, macroscopic phenomena. Usually, we define the ensemble for a many particles system, and then use ensemble average for macroscopic phenomena which represents the system. Also we can define probability density with the ensemble to characterize local properties of the system.

For example, to describe non-relativistic, one dimensional, and one particle Newtonian system with a hamiltonian, $H(P, X)$, a classical phase-space can be chosen for a deterministic representation even though it is skeptical if the representation is complete or not. Once we accept the interaction with vacuum particles, the one particle description in the phase-space is not complete, because non-relativistic Newtonian system is considered as a limiting case when $v/c \ll 1$ in which light velocity(c) never be infinity. By the way, we can understand how non-relativistic Newtonian system is related with interactions with vacuum particles from the relation of universal constants, c and h as[4]

$$\frac{h}{c} = \frac{p_\lambda \lambda}{c} = 2\pi^2 m_e \left(\frac{A^2}{d} \right), \quad (1)$$

where A is the amplitude for each vacuum-particle-string oscillation, d is the equilibrium distance among vacuum particles, and p_λ is the momentum of one wave length in a vacuum-particle-string oscillation. If Plank's constant, $h \rightarrow 0$ or light velocity, $c \rightarrow \infty$ in eqn(1), the amplitude, A on RHS should be zero, thus the interaction effect disappear. But in reality, the effect from the interaction with vacuum particles cannot be zero although it can be ignored in an approximation if it is so small relatively.

2.2 One particle system(1-dim.,time independent Hamiltonian)

Alternatively, we can choose a phase-space with vacuum particles and use classical statistics as a non-deterministic dynamics. In complex space, each real coordinates(\vec{x}) has a 3-dimensional imaginary subspace and interactions of the real physical object with vacuum particles in the subspace follow the physical object in wave forms, that is, infinity number of vacuum-particle-string oscillations. Hence, for one dimensional one particle(real physical object) motion, we can imagine one string-wave corresponded to each real coordinate with momentum, p . Here, *superposition* principle is presupposed to describe the string-wave as

$$\phi(x, p; t) = \frac{1}{\sqrt{2\pi\hbar}} \int \xi(x, x'; t) e^{-\frac{i}{\hbar} p x'} dx', \quad (2)$$

where x' is in the direction of which string-waves propagate in the 3-dimensional imaginary subspace. we might wonder that $\frac{p x'}{\hbar}$ in eqn.(2) is not $\frac{p_\lambda x'}{\hbar}$ since wave number, $k = p_\lambda/\hbar$ in

eqn(1). Because the integration of x' in eqn.(2) is from $-\infty$ to $+\infty$, with a scale transformation of $x' \rightarrow p_\lambda$ can be substituted with p , that is a possible net momentum in the string wave at coordinate x .

$\phi(x, p; t)$ is characterized by $\xi(x, x'; t)$, that is amplitude related with the imaginary coordinate x' in internal complex space (x, ix') , to describe the string-wave corresponded to real coordinate, x . The complex function, $\xi(x, x'; t)$ is not defined in the internal complex space since $\xi(x, x'; t)$ came from a functional relation between the imaginary coordinate, x' in the internal complex space and the momentum, p in the phase space at real coordinate, x . Hence, *analytic* condition is not necessary to the complex function, $\xi(x, x'; t)$. Let us define *internal space*, (x, x') for the function, $\xi(x, x'; t)$.

Since electromagnetic waves was supposed as vacuum-particle-strings oscillation[4], it is natural to assume that $\phi^\dagger(x, p; t)\phi(x, p; t)$ is proportional to energy density or probability density without a loss of generality. Thus, the name, probability amplitude in phase space is appropriate. Let us call $\phi(x, p; t)$ and $\xi(x, x'; t)$ *probability amplitude* and *characteristic amplitude*, respectively.[6] In addition, $\phi(x, p; t)$ is supposed as a continuous function of x and p .

Now we have infinite number of string-waves, each of which is corresponded to a real coordinate. Therefore, an *ensemble* in the phase-space is made up with infinite number of string-waves with a momentum p distribution at coordinate x . Hence, probability density in phase-space can be defined as

$$F(x, p; t) = \phi^\dagger(x, p; t) \phi(x, p; t), \quad (3)$$

in which (x, p) is an abbreviation of continuous and infinite dimensions. And macroscopic phenomena, which is represented with coordinate and momentum of the real physical object, have following relations,

$$\langle p(t) \rangle = \int p(t) F(x, p; t) dx dp \quad (4)$$

and

$$\langle x(t) \rangle = \int x(t) F(x, p; t) dx dp, \quad (5)$$

as *ensemble* average. In a non-relativistic case, we can expect that $\langle p(t) \rangle \sim P(t)$ and $\langle x(t) \rangle \sim X(t)$ as mentioned before. In addition, If we remember that total momentum of string waves is equal to the momentum of real physical object in relativistic or non-relativistic case[4], eqn(4) is self-consistent.

2.3 Wigner-Moyal Infinitesimal Transformation

For any non-relativistic system, the definition of Wigner-Moyal Infinitesimal Transformation[6] is

$$\rho\left(x + \frac{\delta x}{2}, x - \frac{\delta x}{2}; t\right) = \int F(x, p; t) \exp\left(i \frac{p \delta x}{\hbar}\right) dp. \quad (6)$$

This definition is consistent to Wigner Distribution[2] except its infinitesimal nature. From eqn.(2) and eqn.(3), the Infinitesimal Transformation in eqn.(6) can be expressed with a couple of characteristic amplitudes, that is

$$\begin{aligned}\rho\left(x + \frac{\delta x}{2}, x - \frac{\delta x}{2}; t\right) &= \int \xi^*(x, x'; t) \xi(x, x' + \delta x; t) dx' \\ &= \int \xi^*(x, x'; t) e^{\frac{i}{\hbar} \delta x P_{op}} \xi(x, x'; t) dx',\end{aligned}\quad (7)$$

where operator, $P_{op} = -i\hbar \frac{\partial}{\partial x'}$.

In the definition of probability amplitude, $\phi(x, p; t)$, in eqn.(2), momentum p and coordinate x' are related with Fourier transformation. That means, if $\phi(x, p; t)$ is a probability amplitude in phase space (x, p) , then $\xi(x, x'; t)$ is also probability amplitude in internal space, (x, x') . So, we can define the operator, P_{op} as in eqn.(7) to calculate $\langle p^m \rangle$, where m is any integer in principle. With the characteristic amplitude, that is probability amplitude now in internal space (x, x') , the calculation of averages, $\langle x^n \rangle$, $\langle p^m \rangle$, and $\langle x^n p^m \rangle$ or $\langle p^m x^n \rangle$ is straightforward :

$$\langle x^n \rangle = \int \xi^*(x, x'; t) x^n \xi(x, x'; t) dx' dx, \quad (8)$$

$$\langle p^m \rangle = \int \xi^*(x, x'; t) \left(-i\hbar \frac{\partial}{\partial x'}\right)^m \xi(x, x'; t) dx' dx, \quad (9)$$

and

$$\begin{aligned}\langle x^n p^m \rangle &= \langle p^m x^n \rangle \\ &= \int \xi^*(x, x'; t) \left[x^n \left(-i\hbar \frac{\partial}{\partial x'}\right)^m \right] \xi(x, x'; t) dx' dx,\end{aligned}\quad (10)$$

without any ordering problem.

In eqn(7) we can interpret the infinitesimal transformation as an expectation value for infinitesimal translation of $\xi(x, x'; t)$ in the internal space, (x, x') . In addition, from eqn.(6) and eqn.(7), the infinitesimal translation is represented in the phase-space as a symmetry regardless of the sign of δx , but actually it is expressed as a summation of the probability density with infinitesimal oscillations at coordinate, x .

2.4 Complex Function, $\xi(x, x'; t)$ in Internal Space

In former paper[4], we assumed 7 propositions as the starting point. One of them : Any change in imaginary world reflects to real world and *vice versa*. This means *symmetry* and *duality* of Nature.

If complex function $\xi(x, x'; t)$, for example, can be defined in internal space (x, x') , the function, $\xi(x, x'; t)$, must be symmetry under the variable exchange ($x \leftrightarrow x'$). As a possible choice, we can define

$$\xi(x, x'; t) \equiv \psi(x; t) \psi(x'; t). \quad (11)$$

It is a separation of variables and also satisfied with the symmetry. With the definition in eqn.(11) the averages, $\langle x^n p^m \rangle$ and $\langle p^m \rangle$ can be expressed as

$$\langle p^m \rangle = \int \psi^*(x'; t) \left(-i\hbar \frac{\partial}{\partial x'} \right)^m \psi(x'; t) dx', \quad (12)$$

and

$$\langle x^n p^m \rangle = \int \psi^*(x; t) x^n \psi(x; t) dx \int \psi^*(x'; t) \left(-i\hbar \frac{\partial}{\partial x'} \right)^m \psi(x'; t) dx' \quad (13)$$

with $\int \psi^*(x; t) \psi(x; t) dx = 1$.

In internal space, there is no problem in operator ordering since momentum operator, $\hat{p} = -i\hbar \frac{\partial}{\partial x'}$ is defined with coordinate, x' . In addition, if this 2-dimensional formalism is brought to 1-dimensional real space formalism to use only real coordinate x , we cannot find any problem as long as there is no operators *coupled*, such as $\hat{x}^n \hat{p}^m$, $\hat{p}^m \hat{x}^n$, and *etc.*

If a system is described with non-relativistic 1-dimensional Newtonian kinetics for one particle, the energy of the particle,

$$E = \frac{p^2}{2m} + V(x), \quad (14)$$

where m is mass, $V(x)$ is a potential function, of the particle. Using eqn.(12) the average energy,

$$\begin{aligned} \langle E \rangle &= \int \psi^*(x'; t) \left(\frac{\hat{p}^2}{2m} \right) \psi(x'; t) dx' + \int \psi^*(x; t) V(x) \psi(x; t) dx, \\ &= \int \psi^*(x; t) \left[\frac{1}{2m} \left(-i\hbar \frac{\partial}{\partial x} \right)^2 + V(x) \right] \psi(x; t) dx, \end{aligned} \quad (15)$$

where $\hat{p} = -i\hbar \frac{\partial}{\partial x}$.

Now we can express eqn.(14) with operators, $\hat{p} = -i\hbar \frac{\partial}{\partial x}$, and $\hat{x} = x$. That is

$$E \psi(x; t) = \frac{\hat{p}^2}{2m} \psi(x; t) + V(x) \psi(x; t),$$

This is nothing but Schrödinger equation.

In the process, we supposed that the complex function, $\xi(x, x')$ in internal space is separable as $\psi(x)\psi(x')$. But we can choose it as like $\psi^*(x)\psi(x')$ since there is no change in the results. However, we expect *analytic condition* when the function, $\psi(x)\psi(x')$ is mapped from internal space (x, x') to internal complex space (x, ix') . That means, complex function $\psi(x)\psi(ix')$ should be analytic. For instance, the wave function of a free particle is

$$\psi(x) \sim e^{iKx} \quad (K = \frac{P}{\hbar}, P \text{ is momentum})$$

Then $\xi(x, x') \sim e^{iKx} e^{iKx'}$, and through the mapping, that is, $\xi(x, x')$ in $(x, x') \rightarrow \zeta(x, x')$ in (x, ix') ,

$$\begin{aligned} \zeta(x, x') &\sim e^{iKx} e^{-Kx'} \\ &= e^{iK(x+ix')}, \end{aligned}$$

that is analytic. If we choose $\xi(x, x')$ as like $\psi(x)^* \psi(x')$, it corresponds a conventional change in mathematics, such like *right hand rule* or *left hand rule* in physics.

3 Summary

Schrödinger equation(*time independent*) was derived with a classical statistics and complex space model[4]. Even though one dimensional one particle Newtonian system was used, the formalism is general. Therefore, it is straightforward for 3-dimensional and multiparticle system.

We became to know that the probability amplitude in phase space is connected to the wave function in quantum mechanics through internal space (x, x') . Once again we confirmed the physical entity of wave function in quantum mechanics.

It is about time to review those above results with specific cases in detail. Furthermore, once we accept complex space as physical space itself, it is not so hard to figure out the spin of electron in ontological point of view, that is 2-dimensional complex space rotation. Also we, hopely, can understand more closely Pauli exclusion principle, Berry space, and their connection.

References

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